On the Stable Manifold Theorem

Alberto Abbondandolo, Pietro Majer

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Let M be a manifold of class C^r, $1 \leq r \leq \infty$ or $r = \omega$ (as usual, C^ω denotes the analytic category), modeled on the Banach space E. Let $f : M \to M$ be a C^r diffeomorphism. Assume that $\bar{x}$ is a hyperbolic fixed point of f, meaning that $f(\bar{x}) = \bar{x}$ and that there is a $Df(\bar{x})$-invariant splitting $T_{\bar{x}}M = E^s \oplus E^u$ such that, for some equivalent norm $|\cdot|$ on $T_{\bar{x}}M$,

$$|Df(\bar{x})\xi| \leq \lambda|\xi| \quad \text{if } \xi \in E^s, \quad |Df(\bar{x})\xi| \geq \frac{1}{\lambda}|\xi| \quad \text{if } \xi \in E^u,$$

for some $\lambda \in]0,1[$. The definition of hyperbolicity and the subspaces E^s, E^u do not depend on the choice of such a norm. Consider the invariant sets

$$W^s(\bar{x}) = \left\{ x \in M \mid \lim_{n\to+\infty} f^n(x) = \bar{x} \right\}, \quad W^u(\bar{x}) = \left\{ x \in M \mid \lim_{n\to+\infty} f^{-n}(x) = \bar{x} \right\}.$$

The Stable Manifold Theorem asserts the following.

Theorem A *Let M, f and $\bar{x}$ be as above. Then $W^s(\bar{x})$ is an immersed C^r submanifold of M diffeomorphic to E^s. Moreover, $T_{\bar{x}}W^s(\bar{x}) = E^s$.*

By considering f^{-1} instead of f one finds an analogous statement for $W^u(\bar{x})$. Simple examples show that $W^s(\bar{x})$ may not be closed and that it may not be an embedded submanifold.

The usual way to prove such a result is to define the local stable manifold near $\bar{x}$, to prove that in local coordinates such a set is a graph of a Lipschitz map and then to prove further regularity. See, for example, [Shu87], where the graph transformation method as well as Irwin's approach [Irw70] are described with full details. See [Wig94] for many generalizations and for an updated bibliography. Although only elementary tools are needed, detailed proofs are quite long and delicate, so the result is usually stated without proof in many basic textbooks.

The aim of this note is to show a quick proof of Theorem A. The nice feature of the approach presented here is maybe the use of a clear geometric principle, as it is the well-known criterion for local immersion and submersions based on the Inverse Mapping Theorem. This allows to bound the whole analysis to quite a simple linear problem. Moreover, the regularity in the C^r or even in the analytic case follows directly. The same idea works for continuous dynamical systems, i.e. flows obtained by integrating some vector field on M; details are left to the reader.

1 Notations, Definitions and Basic Facts

Linear Operators and splittings. Let $(E, |\cdot|_E)$ and $(F, |\cdot|_F)$ be Banach spaces. We denote by $\mathcal{L}(E,F)$ the Banach space of all linear bounded operators from E to F, endowed with the operator norm $\|T\| := \sup_{|x|_E \leq 1} |Tx|_F$. If $E = F$ we shall simply write $\mathcal{L}(E)$ for $\mathcal{L}(E,E)$. A subspace X (necessarily closed) of F *splits* if and only if there exists a subspace Y such that $F = X \oplus Y$. If $L \in \mathcal{L}(E,F)$ and $R \in \mathcal{L}(F,E)$ are such that $LR = 1_F$, L is called a *left inverse* of R or a *linear retraction* and R is called a *right inverse* of L or a *linear section.* Then L is surjective, R is injective and E splits into a direct sum $E = \ker L \oplus \operatorname{ran} R$, with projections $P_{\operatorname{ran} R} = RL$ and $P_{\ker L} = 1_E - RL$. Conversely, if $R \in \mathcal{L}(E,F)$ is injective and $E = X \oplus \operatorname{ran} R$ for some subspace

X of E, then $L := R^{-1}P_{\operatorname{ran} R} \in \mathcal{L}(E,F)$ is a right inverse of R, with $\ker L = X$. Similarly, if $L \in \mathcal{L}(E,F)$ is surjective and $E = \ker L \oplus Y$ for some subspace Y of E, then $R := (L_{|Y})^{-1}$ is a right inverse of L with $\operatorname{ran} R = Y$.

Immersions and submersions. Let M, N be differentiable manifolds of class C^r, $1 \le r \le \infty$ or $r = \omega$, modeled on the Banach space E, respectively F. A map $f : M \to N$ is a *local immersion* (resp. a *local submersion*), if f is a linear section (resp. a linear retraction) in local charts at x, meaning that there exist a local chart at x, $\varphi : U \to \varphi(U) \subset E$, a local chart at $y := f(x)$, $\psi : V \to \psi(V) \subset F$, and a linear operator $A \in \mathcal{L}(E,F)$ which is a linear section (resp. a linear retraction) and such that $\psi f \varphi^{-1} = A_{|\varphi(U)}$. Then $Y := f(U)$ is submanifold of N and its tangent space at y is $T_yY = \operatorname{ran} Df(x)$, (resp. $X := f^{-1}(V) \cap U$ is a submanifold of M and its tangent space at x is $T_xX = \ker Df(x)$). The map f is said simply an *immersion* (resp. a *submersion*), if it is a local immersion (resp. a local submersion) at any $x \in M$ (resp. $x \in X$). In the first case, if f is also injective, $f(M)$ is said an *immersed submanifold* of N. The usual criterion for local immersion and submersion states that f is a local immersion (resp. a local submersion) at x if and only if $Df(x) \in \mathcal{L}(T_xM, T_yN)$ is a linear section (resp. a linear retraction). A standard reference is [Lan62].

Discrete convolutions on $\ell_{\mathbf{p}}$ classes. If $(E, |\cdot|)$ is a Banach space, the ℓ_p-norm of $u : \mathbf{Z} \to E$ is $|u|_p := \left(\sum_{k\in\mathbf{Z}} |u(k)|^p\right)^{\frac{1}{p}}$, for $1 \le p < \infty$, or $|u|_\infty := \sup_{k\in\mathbf{Z}} |u(k)|$. Then $\ell_p(\mathbf{Z},E)$ denotes the Banach space of all $u : \mathbf{Z} \to E$ such that $|u|_p < \infty$. The set $c_0(\mathbf{Z},E) := \left\{u : \mathbf{Z} \to E \mid \lim_{|k|\to+\infty} = 0\right\}$ is a closed subspace of $\ell_\infty(\mathbf{Z},E)$ and for all $1 \le p \le q < \infty$, $\ell_p(\mathbf{Z},E) \subset \ell_p(\mathbf{Z},E) \subset c_0(\mathbf{Z},E) \subset \ell_\infty(\mathbf{Z},E)$. The analogous class $\{u : \mathbf{N} \to E \mid \lim_{k\to+\infty} = 0\}$ is denoted simply by $c_0(E)$; it can be viewed as a closed splitting subspace of $c_0(\mathbf{Z},E)$. Indeed, there are linear maps $c_0(E) \xrightarrow{j} c_0(\mathbf{Z},E) \xrightarrow{\rho} c_0(E)$, the inclusion j being given by zero-extension, the map ρ being given by restriction to $\mathbf{N} \subset \mathbf{Z}$.

If $g \in \ell_1(\mathbf{Z}, \mathcal{L}(E))$ and $u \in \ell_\infty(\mathbf{Z},E)$, their *convolution product* $g * u$ is defined by $(g*u)(k) := \sum_{j\in\mathbf{Z}} g(k-j)u(j)$. Young's inequality $|g*u|_p \le |g|_1|u|_p$ implies that for any $p \in [1,+\infty]$ the convolution product is continuous as a bilinear map $l_1(\mathbf{Z},\mathcal{L}(E)) \times l_p(\mathbf{Z},E) \to l_p(\mathbf{Z},E)$. Furthermore, $g * u \in c_0(\mathbf{Z},E)$ whenever $g \in \ell_1(\mathbf{Z},\mathcal{L}(E))$ and $u \in c_0(\mathbf{Z},E)$. [1]

Notice that the convolution with $g \in \ell_1(\mathbf{Z},\mathcal{L}(E))$ defines a bounded linear operator R_g on $c_0(E)$ by $u \mapsto g*u$ (more precisely, $R_gu = \rho(g * j(u))$). We shall denote by S the *left shift* $S \in \mathcal{L}(c_0(E))$ defined by $(Su)(k) = u(k+1)$. Finally, we recall that every continuous map $f : E \to F$ with $f(0) = 0$ induces by composition a continuous map $f_* : c_0(E) \to c_0(F)$, $f_*(u) := f \circ u$. If f is of class C^r, $1 \le r \le +\infty$ or $r = \omega$, so is f_*, and the j-th differential at $u \in c_0(E)$ is given by the formula $\left(D^jf_*(u)[v]^j\right)(k) = D^jf(u(k))[v(k)]^j$.

2 The Stable Manifold Theorem

We may assume that $M = E$ and that $\bar{x} = 0$: the case of a general manifold can be easily deduced by "topological continuation[2] ", like in the proof of Theorem 2.2. So $f : E \to E$ is a diffeomorphism of class C^r, $1 \le r \le \infty$ or $r = \omega$, $f(0) = 0$ and $Df(0) =: T \in \mathcal{L}(E)$ is *linear hyperbolic*, i.e. there exists a T-invariant splitting $E = E^s \oplus E^u$ such that

$$|Tx| \le \lambda|x| \quad \text{if } x \in E^s, \quad |T^{-1}x| \le \lambda|x| \quad \text{if } x \in E^u,$$

for some $\lambda \in]0,1[$. We wish to prove that

$$W^s(0) := \left\{x \in E \mid \lim_{n\to+\infty} f^n(x) = 0\right\}$$

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[1] This follows immediately by approximating g with the sequence $g_n := \chi_{[-n,+n]}g$, for $g_n \to g$ in ℓ_1, $g_n * u \in c_0(\mathbf{Z},E)$ and by Young's inequality $|g*u - g_n*u|_\infty = |(g-g_n)*u|_\infty \le |g_n - g|_1|u|_\infty \to 0$, so $g*u = \lim_{n\to\infty} g_n*u \in c_0(\mathbf{Z},E)$.

[2] We borrow the expression from [Sma67].

is an immersed C^r submanifold of E, diffeomorphic to E^s and such that $T_0 W^s(0) = E^s$. Set

$$\mathcal{M} := \{u \in c_0(E) \mid u(n+1) = f(u(n)) \ \forall n \in \mathbf{N}\}.$$

Therefore, $W^s(0) = \{u(0) \mid u \in \mathcal{M}\}$. Our strategy is to use the Implicit Function Theorem to prove that $\mathcal{M}$ is an embedded submanifold of $c_0(E)$, and then to show that the evaluation map $\mathcal{M} \ni u \mapsto u(0) \in E$ is an injective immersion. Notice that $\mathcal{M} = (S - f_*)^{-1}(\{0\})$, where S and f_* have been described above. We start by studying the differential of the map $S - f_*$ at 0, which is $D(S - f_*)(0) = S - T_*$.

Lemma 2.1 *Assume that $T \in \mathcal{L}(E)$ is linear hyperbolic with constant $\lambda \in]0,1[$. Set, for $k \in \mathbf{Z}$,*

$$g(k) := T^{k-1}\left(\chi_{\mathbf{Z}^+}(k) 1_E - P_{E^u}\right),$$

where $\mathbf{Z}^+ = \{1, 2, \dots\}$. Then $g \in \ell_1(\mathbf{Z}, \mathcal{L}(E))$ and the corresponding convolution operator $R_g \in \mathcal{L}(c_0(E))$ is a right inverse of $S - T_*$. Moreover,

$$\ker(S - T_*) = \left\{u \in c_0(E) \mid u(0) \in E^s, \ u(k) = T^k u(0) \ \forall k \in \mathbf{N}\right\}.$$

Proof. By the hypotheses, $|g(k)| = |T^{k-1} P_{E^s}| \leq \lambda^{k-1}$ if $k > 0$, while $|g(k)| = |T^{k-1} P_{E^u}| \leq \lambda^{-k+1}$ if $k \leq 0$, so $g \in \ell_1(\mathbf{Z}, L(E))$, since $|g|_1 = \sum_{k \geq 1} \lambda^{k-1} + \sum_{k \geq 0} \lambda^{k+1} = \frac{1+\lambda}{1-\lambda} < +\infty$. We have, for any $u \in c_0(E)$ and $k \in \mathbf{N}$,

$$[(S - T_*) R_g(u)](k) = \sum_{j=0}^{+\infty} g(k+1-j) u(j) - \sum_{j=0}^{+\infty} T g(k-j) u(j)$$

$$= \sum_{j=0}^{+\infty} T^k \left[\chi_{\mathbf{Z}^+}(k+1-j) - \chi_{\mathbf{Z}^+}(k-j)\right] u(j) = \sum_{j=0}^{+\infty} T^k \chi_{\{0\}}(k-j) u(j) = u(k),$$

that is, $(S - T_*) R_g = 1_E$. [3] Finally, it is clear that $u \in \ker(S - T_*)$ if and only if $u(k) = T^k u(0)$ for any $k \in \mathbf{N}$, which defines an element of $c_0(E)$ if and only if $u(0) \in E^s$. □

Notice that the closed space

$$\operatorname{ran} R_g = \{u \in c_0(E) \mid u(0) \in E^u\}$$

is a direct summand of $\ker(S - T_*)$, the projections being $P_{\ker(S - T_*)} = [S - T_*, R_g]$, which maps u to $\left(T^k P_{E^s} u(0)\right)_{k \geq 0}$, and $P_{\operatorname{ran} R_g} = R_g(S - T_*) = 1_E - P_{\ker(S - T_*)}$.

Theorem 2.2 *If f is C^r, $1 \leq r \leq +\infty$ or $r = \omega$, the map $S - f_*$ is a local submersion at any point $x \in \mathcal{M}$, and $\mathcal{M}$ is a C^r submanifold of $c_0(E)$, diffeomorphic to E^s, such that $T_0\mathcal{M} = \ker(S - T_*)$.*

Proof. We have $D(S - f_*)(0) = S - T_*$, which is a linear retraction by Lemma 2.1. Thus $S - f_*$ is a local submersion at some open neighborhood U of 0. If $u \in \mathcal{M}$, for some $m \in \mathbf{N}$ the m-th iterated f_*^m subordinates a diffeomorphism of some open neighborhood V of x into U, $f^m_{*|V} : V \to f_*^m(V) \subset U$, and $f_*^m(V \cap \mathcal{M}) = f_*^m(V) \cap \mathcal{M}$. This shows that $S - f_*$ is a submersion at any point of $\mathcal{M}$, and $\mathcal{M}$ is a submanifold of $c_0(E)$, such that $T_0\mathcal{M} = \ker(S - T_*)$. □

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[3] Here is a more heuristic argument to find a left inverse to the linear operator $S - T_*$. First notice that the equation $(S - T_*) = h \in c_0(E)$ is equivalent to $u(k+1) = Tu(k) + h(k)$, $\forall k \geq 0$, that iterated gives $u(k) = T^k u(0) + \sum_{j=0}^{k-1} T^{k-1-j} h(j)$. We can split this equation into $u(k) = T^k P_{E^s} u(0) + \sum_{j=0}^{k-1} T^{k-1-j} P_{E^s} h(j) + T^k \left[P_{E^u} u(0) + \sum_{j=0}^{k-1} T^{-1-j} P_{E^u} h(j)\right]$. Now the first and the second term converge as $k \to \infty$ since $\|T P_{E^s}\| \leq \lambda$. The third term may not converge unless the sequence into square brackets converge to 0, that is, $P_{E^u} u(0) + \sum_{j=0}^{k-1} T^{-1-j} P_{E^u} h(j) = -\sum_{j=k}^{+\infty} T^{-1-j} P_{E^u} h(j)$, whence $u(k) = T^k P_{E^s} u(0) + (g * h)(k)$.

Remark 2.1 *More is true: the map $S - f_*$ is a submersion on the whole domain $c_0(E)$; however we shall not need this stronger fact.*

Theorem 2.3 *The evaluation operator $ev_0 \in \mathcal{L}(c_0(E), E)$ subordinates an injective immersion of $\mathcal{M}$ onto $W^s(0)$, and $W^s(0)$ is an immersed submanifold of E with $T_0 W^s(0) = E^s$.*

Proof. The map ev_0 is indeed injective, for $u \in \mathcal{M}$ if and only if $u(k) = f^k u(0)$, $\forall k \geq 0$. Similarly, $D(ev_{0|\mathcal{M}})$ is injective, for $v \in T_0\mathcal{M} \subset c_0(E)$ if and only if $v(k) = T^k v(0)$, $\forall k \geq 0$. Moreover, $\operatorname{ran} D(ev_{0|\mathcal{M}}) = E^s$, a splitting subspace of E by hypothesis.

The map ev_0 is therefore a local immersion at 0; thanks to the same "topological continuation argument" used in Theorem 2.2, it turns out to be an injective immersion of $\mathcal{M}$ onto $W^s(0)$. Hence, $W^s(0)$ is an immersed submanifold of E. Finally, $T_0 W^s(0) = \operatorname{ran} D(ev_0|\mathcal{M}) = \operatorname{ran}\left(ev_{0|T_0\mathcal{M}}\right) = E^s$, just because $T_0\mathcal{M} = \ker(S - T_*)$. $\square$

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